Josephus Permutations

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Introduction

The ‘Josephus problem’ refers to a famous puzzle in mathematics and computer science related to counting. Back in the day, a Jewish commander named Flavius Josephus was trapped in a cave with 40 of his soldiers by the Roman troops. Instead of surrendering and being captured, they decided to kill each other in turns (and the last surviving person would commit suicide). The rules of this (very grim) ritual were as follows:

> Josephus and 40 of his soldiers were arranged in a circle. After picking a starting position, everyone would kill the third person alive next to him, proceeding clockwise. This pattern continues until there is only one survivor in the circle, who would commit suicide. **[6]**

However, Josephus and one of his soldiers would rather be captured and stay alive than be killed. Afraid of causing a riot, they had to follow this game. Their only chance of surviving was to figure out which are the two positions of the last survivors so they could both surrender to the Roman troops after everyone else was dead.

Although this story has become a favorite word problem in mathematics classrooms, many current curricular activities related to the Josephus problem only focus on computing the position of the last survivor. This mainly involves solving a recurrence relation, where we would assume there are n people in the circle and every kth person is getting killed. In this project, we will look at the Josephus problem in a different way. Naturally, the Josephus problem determines a sequence of numbers – specifically, the sequence, or order, in which people are killed. We can consider this sequence as a permutation of [n]. In this project, I will design a lesson plan with the purpose of strengthening the students’ understanding of permutations using this interesting counting game.

Students will start by getting familiar with the game. Then they will define these “Josephus functions” themselves, in which the formal definition of functions and some properties will be introduced. Finally, students will explore whether the Josephus permutations form a subgroup of the associated symmetric group. The lesson plan focuses on undergraduate level math, especially in discrete math and group theory. Since the concepts of sequences and permutations are crucial in upper-level math courses like analysis and algebra, I think this lesson plan, with its interesting counting game, will help students strengthen their understanding of functions and provide them with an application of permutation groups. The first two parts of the lesson plan – those that deal with the Josephus problem and the Josephus functions – can be implemented in a discrete math course like MTH 356. The later topics – concerning permutations – fit most naturally in a group theory course like MTH 344. The overall goals of the curriculum are to introduce the formal definition of functions/sequences, as well as some properties, like surjections, injections, and bijections using the context of the Josephus problem as an example.

The Josephus Problem

Josephus problem is a well-known theoretical problem in the field of mathematics and computer science. The problem was named after Titus Flavius Josephus, a Jewish historian in the first century A.D. He was a commander in the first Jewish-Roman war. In his publication *The Works of Flavius Josephus: in three volumes; with illustration* **[6]**, there is a story that describes what is known as the Josephus problem today. The story is as such:

> In a battle with the Romans, Josephus and his 40 comrades were trapped in a cave by the Roman troops. In this situation, all the comrades would rather die than be slaves to the Romans. Hence, they chose to suicide over being captured. At last, they settled on a set of rules to kill each other as follows:
>
> They arranged themselves in a circle. One man was designated as number one, and they proceeded clockwise around the circle, killing every 3rd person alive next to him. The last person alive would commit suicide so no one would be captured.
>
> However, Josephus and one of his comrades would rather join the Romans and stay alive. Afraid of causing a riot, they had to play along. Their only chance of surviving was to figure out the position of the last two survivors so they can both surrender after everyone else was dead.

Notice that there are three variables in this problem: (1) the number of people participating in this killing game, (2) the position of the person getting killed, and (3) the number of survivors at the end of the game. We will discuss these three variables in detail.

(1) Number of People

This is a straightforward variable. We will define n as the number of people participating in the killing game. Clearly, $n \in \mathbb{N}$. For example, if there are 4 people participating in the killing game, then $n = 4$ and these 4 people will be arranged in a circle labeled from 1 to 4.

(2) Position of the First Person to be Killed

This variable will be the most confusing among the three. In some texts, this is called the **skip number**. We will define k as the skip number. The skip number is defined as the position of first person to be killed, counting the person who do the killing. For example, if $k = 2$, then the second person will be the first person killed. As the process continues, every other (alternatingly around the circle) person will be killed. If $k = 3$, then the person third person will be the first person killed, and every third person after that will be killed in turn. Clearly, $k \in \mathbb{N}$. Also, it can be ambiguous in cases where $n < k$. We will consider this situation later, and we will propose a solution to clear up the ambiguity.

(3) Position of the survivors

Now that we have defined the first two variables, we can see that they are the natural choices for independent variables. Indeed, we want to compute the position of the last survivors in a circle of n people with every kth person getting killed. We will define j as such the position. This is a dependent variable, depending on n and k. Hence, we will notate it as $j_{n,k}$ .The goal is to come up with a formula that will compute $j_{n,k}$ given the values of n and k.

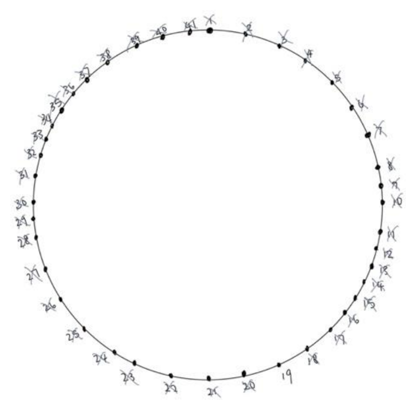
Simplified Josephus Problem

Before we discuss the Josephus problem, we will discuss a simplified version of the problem. Having n fixed, we let $k = 2$, which means every other person is killed in the circle and we will solve for $j_{n,2}$, which is the position of the last survivor in the circle. To simplify the notation, we will denote $j(n) = j_{n,2}$ in this section. There are several ways to determine $j(n)$ and we will solve for $j(41)$ just to model it as close to the original Josephus problem as possible.

Brute Force

Now that we know the rules of the game, one method would be to just brute force the solution by drawing a circle with $n = 41$ and eliminating people until there is only one left.

The circle of elimination is shown below:

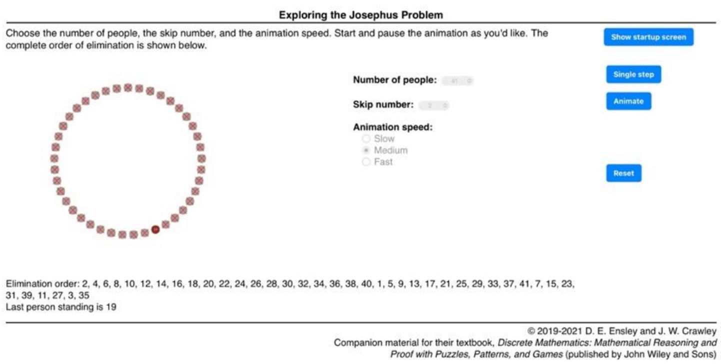


If we follow the rule and kill every second person, the position of the last survivor will be 19.

There is an observation worth noticing here. After the first round of elimination, all the even numbers are eliminated. This implies that the solution could NOT be an even number. By the parity of integers, the solution should be an odd integer.

With the Help of the Technology

There are some online programs and applets that model the Josephus problem, and they will solve for the position of the last survivor when the number of people is plugged in. There is an applet shown below to demonstrate the solution of $j(41)$:



(webpage: https://webspace.ship.edu/jwcraw/dmrev2dev/Chapter1/1-1-Josephus.html)

In this applet, we can also see that the position of the last survivor is position 19. Also, the order of elimination is provided. If we want to know the position of the last two survivors, they will be 19 and 35. Also, notice that the value of k can be adjusted in the applet as well, and we will discuss the situation where $k \geq 3$ later.

Pattern Noticing

We can also solve the problem by noticing patterns in some small cases. We can create the following table for values of n up to 9:

n	1	2	3	4	5	6	7	8	9
$j(n)$	1	1	3	1	3	5	7	1	3

There are a few observations worth noticing in the table:

1. When $n = 2^m$ for some $m \in \mathbb{Z}$, then $j(n) = 1$;

2. When $n = 2^m + l$ for some $l \in [0,2^m)$, $j(n)$ is an odd number and $j(n) = 2l + 1$.

These observations are actually sufficient for us to find the solution of the problem. By the division algorithm, we know that $n = 41 = 2^5 + 9$. Then, we let $l = 9$ and

$$j(n) = 2(9) + 1 = 19 \,.$$

Using this method, we can generate the solution of an infinite family of the simplified Josephus problems. Specifically, if n is given, we can compute $j(n)$ by identifying the values of m and l. Notice that there are three variables in the formula (n, m, and l). It is not user friendly, as we need to decompose an integer to identify three variables. If we were able to deduce the formula to just one variable, then we could get the solution faster. Notice that m and l are determined once n is picked. Hence, there is only one true independent variable, n. If we can express the formula solely in terms of n, then the formula will become explicit. There are two variables we need to eliminate.

Our first step is to combine two formulas into one. Notice that l appears in both formulas, so we can make a substitution and we obtain

$$j(n) = 2(n - 2^m\ ) + 1 \ .$$

Our next step is to eliminate m. By the division algorithm, we know that

$$n = 2^m + l \ .$$

Solving for m, we obtain

$$m = \log_2(n - l) \ .$$

Now we can eliminate m and the formula becomes:

$$j(n) = 2\left(n - 2^{\log_2(n-l)}\right) + 1 \ .$$

Notice that we introduced l back into the formula. The final step is to eliminate this l without introducing any new variables. Remember that $m = \log_2(n-l)$ and $m \in \mathbb{N} \cup \{0\}$. We can conjecture that reducing l indicates that we need to introduce some rounding functions. Notice that $n - l < n$. Then

$$m = \log_2(n-l) < \log_2 n \ ,$$

Since the logarithmic function with base 2 is an increasing function. Also, by the division algorithm, $n < 2^{m+1}$. Then

$$m = \log_2(n-l) < \log_2 n < \log_2 2^{m+1} = m + 1 \ .$$

Remember that we need to round $\log_2(n-l)$ so that this value equals to m. Clearly, we need to round down $\log_2(n-l)$ so it equals m. Hence, we take the floor function, and we will obtain

$$m = \lfloor \log_2 n \rfloor < \log_2(n-l) \ .$$

Notice that there is no l and m anymore.

Therefore, the formula $j(n) = 2\left(n - 2^{\lfloor \log_2 n \rfloor}\right) + 1$ is an explicit formula solely in terms of n. To verify that we did our work right, we can compute $j(41)$ and see whether we will get 19 as the output. Notice that

$$j(41) = 2\left(41 - 2^{\lfloor \log_2 41 \rfloor}\right) + 1 = 19 \ .$$

We now obtain an explicit formula, and we can compute solutions to a family of simplified Josephus problems with it.

Binary Number Pattern

If we convert n and $j(n)$, there is an interesting phenomenon. The table below shows the values of n and $j(n)$ in decimal and binary:

$(n)_{10}$[1]	1	2	3	4	5	6	7	8	9
$\left(j(n)\right)_{10}$	1	1	3	1	3	5	7	1	3
$(n)_2$	1	10	11	100	101	110	111	1000	1001
$\left(j(n)\right)_2$	1	01	11	001	011	101	111	0001	0011

Notice that if the leading 1 of n is moved to the last place, this is the $j(n)$. We conjecture that we can obtain our $j(n)$ in binary if we move the leading 1 in n to the last place. If we can show that the conjecture is true, then we will have an alternative approach for solving the family of the simplified Josephus problem. In other words, to find $j(n)$, all we have to do is to convert n in binary, move the leading 1 to the last place, and convert this number back to decimal. This is our $j(n)$.

In fact, this is the same approach as the previous one, but has been "translated" into binary from decimal. Consider our piecewise formula for $j(n)$:

If $n = 2^m$, then $j(n) = 1$. We know that $(n)_2 = 100 \ldots 000$ with m zeros. Moving the leading 1 to the last place will result in $j(n) = 000 \ldots 001 = 1$.

If $n = 2^m + l$, then $(n)_2 = (2^m + l)_2 = (2^m)_2 + (l)_2$. We know that $j(n) = 2l + 1$. Then

$$\begin{aligned}\left(j(n)\right)_2 &= (2l+1)_2 \\ &= (2l)_2 + 1 \\ &= (l+l)_2 + 1 \\ &= (l)_2 + (l)_2 + 1 \\ &= 2(l)_2 + 1\ .\end{aligned}$$

[1] We will use $(n)_{10}$ to denote n in decimal number system and $(n)_2$ to denote n in binary number system.

Notice that $(l)_2 = (n)_2 - (2^m)_2$. Then by substitution, we obtain

$$\big(j(n)\big)_2 = 2((n)_2 - (2^m)_2) + 1 \ .$$

Notice that subtracting $(2^m\ )_2$ causes us to remove the leading 1 in $(n)_2$. Multiplying by 2 causes us to add a 0 at the end, and adding one causes us to switch the ending 0 to 1. This shows why the method of moving the leading 1 to the last place in binary works to solve for the position of the last survivor.

Now that we have an alternative approach, we will compute $j(41)$. Notice that 41 can be decomposed as follows:

$$41 = 1\ \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 \ .$$

Hence, the binary of 41 is 101001.

Using the binary number pattern, we know that

$$\begin{aligned} j(41) &= (010011)_{10} \\ &= 0 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \\ &= 16 + 2 + 2 = 19\,. \end{aligned}$$

We get $j(41) = 19$, and this agrees with the solution given by previous approaches.

Actual Josephus Problem

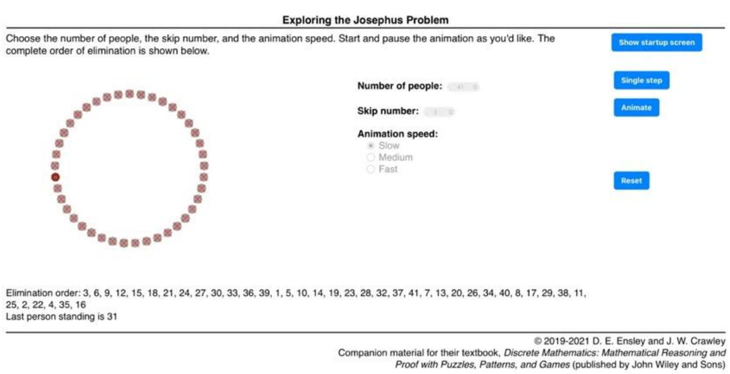
In the actual Josephus problem, we are solving for $j_{41,3}$ (and the second to last spot, but we will focus on the number of the last survivor). In this section, we will assume that $k = 3$ and $j(n)$ denotes $j_{n,3}$. We want to compute $j(41)$. We will look at several approaches for the actual Josephus problem.

Brute Force

This is always an available approach to solve the Josephus problem, once the rule of the game is established. Using brute force, we can get that $j_{41,3} = 31$. Hence, Josephus would want to stand in spot #31. However, the process gets very difficult as n and k get larger.

With the Help of the Technology

Thanks to modern technology, there are ready-to-use applets online that can solve the problem right away. The applet introduced in the last section can also be applied to solve the problem, shown below:



Notice that the applet gives the position of the last survivor, which is $j_{41,3} = 31$.

Maybe an Explicit Formula?

Recall that in the simplified Josephus problem, we were able to deduce an explicit formula, where we can compute $j(n)$ by simply plugging in n into the formula. However, this approach is not practical here because there is no obvious pattern in the smaller cases when $k = 3$. Let us take a look at the position of the last survivors in small cases (see below).

n	1	2	3	4	5	6	7	8	9
$j(n)$	1	2	2	1	4	1	4	7	1

It is difficult to see any pattern by looking at this small data set. However, if we consider the rule of the game, we can see that every time a person is killed, we are creating a smaller circle with the same rule. This is a hint that we might be able to come up with a recurrence relation, as we can create smaller cases related to the original case.

Recurrence Relation for the Last Survivor

The goal of this section is to come up with a recurrence relation of $j_{n,3}$ in terms of n.

Imagine there is a circle with n people labeled from 1 to n clockwise, where $n \in \mathbb{N}$. After eliminating one person from the circle, the total number of people in the circle is $n - 1$, and the rule remains the same. In other words, if we consider the next person alive as the starting point, the position of the last survivor in the circle of $n - 1$ people will be the same as that in the circle of n people, as nothing has changed but the labels of people. Hence, we know that if we relabel, we can obtain a recurrence relation where $j(n)$ is determined by $j(n - 1)$.

Let us consider the relabeling. In a circle of n people where position 1 starts the killing, the person in position 3 will be killed and the next person alive is in position 4, who will be the person starting the killing in a circle of $n - 1$ people. In other words, this will be the position 1 in the circle of $n - 1$ people. Hence, the difference of labeling between these two cases is 3. Since the rule of the game remains the same in the two cases, $j(n) = j(n - 1) + 3$ since, shifting the labels clockwise by 3 units causes the position of the last survivor to also get shifted by 3 units, even though they are the same person. Now we have a recurrence relation. But we need to consider the nature of the addition operation. As we get fewer people in the circle, we will keep adding 3 in order to adjust our labeling. At some point, the label will exceed n because $\mathbb{Z}$ under addition is an infinite group. We cannot have a label greater than n, as n is the number of people we started with. To fix this issue, we need to introduce modular arithmetic to ground the label no more than n. Hence, the recurrence relation becomes

$$j(n) = (j(n-1) + 3) \bmod n$$

as we don't want any label greater than n.

This causes another issue. Consider the output of modular arithmetic. By the division algorithm, $p = nq + r$ for some $p, n, q, r \in \mathbb{Z}$ and $r \in [0, n)$. Hence, we know that $p \bmod n = r$, where $r \in [0, n)$. Looking back at our recurrence relation, we don't have label 0, as we start our labeling in 1, and we can have label n if position n happens to be the position of the last survivor.

In order to fix this issue, we need to create a "0" label by subtracting 1 in the relabeling process. Then the position of the last survivor will be $(j(n-1) + 3 - 1) \bmod n$. The difference in the labeling of two cases will be 1, so we just need to adjust the labeling by adding 1 back at the end. Hence, we now have our recurrence relation:

$$j(n) = ([j(n-1) + 2] \bmod n) + 1 \ .$$

In order to apply this relation, we need a base case. The smallest number of people in a circle is 1, and if there is only 1 person in a circle, by default, they will be the last survivor. Hence, we have $j(1) = 1$. Now we can compute the last survivor when $n = 41$. An Excel function can be used to compute $j(41)$ and the solution is $j(41) = 31$. This implies that if Josephus stands in position 31, he is guaranteed to be the last survivor.

Generalized Josephus Problem

In the generalized Josephus problem, there are n people in a circle and every kth person is killed. We are still interested in knowing the position of the last survivor. Hence, we want to compute $j_{n,k}$ with arbitrary n and k. The strategy of how to come up with a recurrence relation for the actual Josephus problem applies here. If we replace 3 by k from the last section, then the recurrence relation becomes:

$$j_{n,k} = ([j(n-1) + k - 1] \bmod n) + 1 \ .$$

We know that once k is fixed, then we can only change n, as you cannot change the rule of the game. Hence, k is a constant once it is determined. Also, we saw how we can manipulate n to smaller cases in the last section. Since the same strategy works for other values of k, the final formula will hold for all $n, k \in \mathbb{N}$.

The Josephus Permutations

For a Josephus problem with $n, k \in \mathbb{N}$, it is obvious that there is a sequence of people being killed. We can define a permutation from $\{1,2,\dots,n\}$ to the sequence of people being killed. We will denote by $J_{n,k}$ a permutation produced by the Josephus problem for a fixed $n \in \mathbb{N}$ and $k \in \mathbb{N}$. We can make a few observations:

1. As stated above, if we fix both $n \in \mathbb{N}$ and $k \in \mathbb{N}$, a unique Josephus permutation will be produced.

2. If we fix $k \in \mathbb{N}$, then there are infinitely many Josephus permutations, as each $n \in \mathbb{N}$ will produce a unique Josephus permutation.

3. If we fix $n \in \mathbb{N}$, then we are permuting natural numbers 1 to n by the rule of the Josephus problem. The Josephus permutation produced is an element in the symmetric group S_n. We know that S_n is a finite group. This implies that there are finitely many Josephus permutations if we fix $n \in \mathbb{N}$.

4. If we fix neither $n \in \mathbb{N}$ nor $k \in \mathbb{N}$, similar to observation 2, then there are infinitely many Josephus permutations.

Considering observation 3, we define $\mathbb{J}_n$ to be the set of all Josephus permutations of size n. Then, by inspection, $\mathbb{J}_n \subseteq S_n$. In this section, we will look at the problem of whether $\mathbb{J}_n$ forms a group for any arbitrary $n \in \mathbb{N}$.

Number of Josephus Permutations

Notice that $\mathbb{J}_n \subseteq S_n$. Then $|\mathbb{J}_n| \leq |S_n|$. Before we dive into determining $|\mathbb{J}_n|$, we need to decide on whether there is any restriction of $k \in \mathbb{N}$. It is obvious that the Josephus problem makes sense if $k < n$.

Consider the case where $k \geq n$. Notice that the Josephus problem operates on a circle. Every time $k \geq n$, we can go over the circle and modular arithmetic will be used to reduce any larger number than n to a unique number from 1 to n. Hence, the Josephus problem also makes sense when $n \geq k$. This implies that there is no restriction on k. For a fixed $n \in \mathbb{N}$, k can be any natural number.

As stated in the previous observation, $\mathbb{J}_n$ is a finite set but there are infinitely many values of k. This implies that there must be multiple values of k that will produce the same Josephus permutation. By inspection, if we spin around the circle enough times, we will produce all the distinct Josephus permutations in $\mathbb{J}_n$. We make the following claim:

Proposition: Let $n \in \mathbb{N}$. Then $|\mathbb{J}_n| = lcm\{1,2, \dots, n\}$.

Proof: Let $l(n) = lcm\{1,2, \dots, n\}$. Let $k \in \{1, \dots, n\}$ and $k' = k + l(n)$. Notice that $l(n) \equiv 0 \pmod{k}$ for all $k \in \{1, \dots, n\}$. So $J_{n,k}$ and $J_{n,k'}$ will agree on every step. This implies that $|\mathbb{J}_n| \leq l(n)$.

Now fix $k_1, k_2 \in \{1,2, \dots, l(n)\}$ such that k_1 and k_2 produce the same Josephus permutation. Then k_1 and k_2 will satisfy the same constraints for each step of the Josephus problem. This implies that k_1 and k_2 will have the same factors. By the unique factorization of integers, $k_1 = k_2$. This implies that each $k \in \{1,2, \dots, l(n)\}$ will produce a unique Josephus permutation. Hence, $|\mathbb{J}_n| \geq l(n)$.

Therefore, $|\mathbb{J}_n| = l(n) = lcm\{1,2, \dots, n\}$. ■

Now we have determined the size of $\mathbb{J}_n$. Clearly, $\mathbb{J}_n$ is a non-empty and finite set for any $n \in \mathbb{N}$. Also, we have established the fact that $\mathbb{J}_n \subseteq S_n$ for all $n \in \mathbb{N}$. If $\mathbb{J}_n$ is a group for some $n \in \mathbb{N}$, then $\mathbb{J}_n$ must be a subgroup of S_n. So we can determine whether $\mathbb{J}_n$ is a group by using the finite subgroup test.

Finite Subgroup Test: Let H be a non-empty finite subset of a group G. Then H is a subgroup if and only if H is closed under the operation.

Proof: Suppose $H \leq G$. Then by the definition of subgroups, H is a group, and hence H is closed.

Conversely, suppose H is closed under the operation. Let $h \in H$. Then $h^k \in H$ for all $k \in \mathbb{N}$ by closure. Consider the infinite list $h, h^2, h^3, \dots$. By closure, $h, h^2, h^3, \dots \in H$. Since H is finite, then there must be repeats. WLOG, let $i, j \in \mathbb{N}$ with $i < j$ be such that $h^i = h^j$. This implies that

$$e = h^i h^{-i} = h^j h^{-i} = h^{j-i} .$$

This implies that $e \in H$, since $j - i > 0$. Notice that $h^{j-i} = hh^{j-i-1} = h^{j-i-1}h$. By substitution, we have $e = h^{j-i} = hh^{j-i-1} = h^{j-i-1}h$. Also, $j - i - 1 \geq 0$. Then $h^{-1} = h^{j-i-1} \in H$. Since $h \in H$ is arbitrarily chosen, then we showed that each element in H has an inverse in H.

We showed that H has an identity and each element in H has an inverse in H, in addition to H being closed. By the full subgroup test, H is a subgroup of G.

$\therefore H \leq G$ if and only if H is closed under the operation. ■

Using the finite subgroup test, all we need to know is whether $\mathbb{J}_n$ is closed under composition. If so, then by the finite subgroup test $\mathbb{J}_n$ is a group; if not, $\mathbb{J}_n$ is not a group, as it is not closed.

Another observation we can make before classifying $\mathbb{J}_n$ is to consider the sizes of $\mathbb{J}_n$ and S_n. We showed that $\mathbb{J}_n = lcm[1,2,\ldots,n]$ and $S_n = n!$. We will compute the sizes of them for $n = 1,2,3,4,5,6,7$ in the following table:

n	1	2	3	4	5	6	7
$\lvert\mathbb{J}_n\rvert$	1	2	6	12	60	60	420
$\lvert S_n\rvert$	1	2	6	24	120	720	5040

Based on the values of $|\mathbb{J}_n|$ and $|S_n|$, we can make the following three observations:

1. for $n \in \{1,2,3\}$, $|\mathbb{J}_n| = |S_n|$;

2. for $n \in \{4,5\}$, $|\mathbb{J}_n| = \frac{|S_n|}{2}$;

3. for $n \geq 6$, $|\mathbb{J}_n| < \frac{|S_n|}{2}$.

We will develop the following section of classifying $\mathbb{J}_n$ for all $n \in \mathbb{N}$ using these observations.

Classification of Josephus Permutations for $\boldsymbol{n} \in \mathbb{N}$

We will look at each $\mathbb{J}_n$ and decide whether $\mathbb{J}_n$ forms a group under permutation composition.

Suppose $n = 1$. By observation 1, $|\mathbb{J}_1| = |S_1|$. Also, $\mathbb{J}_1 \subseteq S_1$. This implies that $\mathbb{J}_1 = S_1$ as a set. We know that $S_1 = \{(1)\}$ where (1) represents the identity permutation. Then $\mathbb{J}_1 = \{(1)\}$. Clearly, (1) is the Josephus permutation of $n = 1$ and $k = 1$. Then $\mathbb{J}_1 = S_1$ is a group.

Suppose $n = 2$*.* By observation 1, $|\mathbb{J}_2| = |S_2|$. Also, $\mathbb{J}_2 \subseteq S_2$. This implies that $\mathbb{J}_2 = S_2$ as a set. Since $\mathbb{J}_2$ and S_2 are equipped with the same binary operation, which is function composition, then we can conclude that $\mathbb{J}_2 = S_2$ as a group.

Suppose $n = 3$*.* Similar to the case where $n = 2$, we can conclude that $\mathbb{J}_3 = S_3$ as a group.

Suppose $n = 4$*.* By observation 2, $|\mathbb{J}_4| = \frac{|S_4|}{2} = 12$. This is a small set, and we can write out every element in $\mathbb{J}_4$. All elements in $\mathbb{J}_4$ is listed in **Appendix A**. We will define two terms as follows:

Definition 1: An **even** permutation is a permutation that can be expressed as a composition of an even number of transpositions.

Definition 2: The subgroup of S_n consisting of the even permutation of n letters is called the **alternating group** A_n on n letters.

Notice that all Josephus permutations in $\mathbb{J}_4$ are even permutations. Also, $\mathbb{J}_4 \subseteq S_4$. Hence, $\mathbb{J}_4 = A_4$ as a set. Based on Definition 2, A_4 is a group. Hence, $\mathbb{J}_4 = A_4$ as a group.

Suppose $n = 5$*.* By observation 2, $|\mathbb{J}_5| = \frac{|S_5|}{2} = 60$. We have the same pattern as the case of $n = 4$, so we can make the conjecture that $\mathbb{J}_5 = A_5$. To show that our conjecture is true, a method is to list out all elements in $\mathbb{J}_5$ and see if all Josephus permutations in $\mathbb{J}_5$ are even. All elements of $\mathbb{J}_5$ are listed in **Appendix B**. By inspection, we can see that $\mathbb{J}_5$ consists of all even permutations of 5 letters. This implies that $\mathbb{J}_5 = A_5$ as a set. Hence, $\mathbb{J}_5 = A_5$ as a group.

Before we continue on to the case where $n \geq 6$, by observation 3, we can no longer make the conjecture that $\mathbb{J}_n = A_n$, as $|\mathbb{J}_n| < \frac{|S_n|}{2}$. Yet, notice that $0 < |\mathbb{J}_n| < \infty$. By the finite subgroup test, to show that $\mathbb{J}_n$ is a group (and hence a subgroup of S_n), we only have to show that $\mathbb{J}_n$ is closed under composition. On the other hand, if $\mathbb{J}_n$ is not closed under composition, then it violates the definition of a binary operation, and hence $\mathbb{J}_n$ is not a group. That is, we only need to know whether $\mathbb{J}_n$ is closed in order to know whether $\mathbb{J}_n$ is a group.

We claim that $\mathbb{J}_n$ is not a group for $n \geq 6$. Before we prove this claim, we will look at the following two useful propositions in number theory:

Proposition 1: The sum of two integers with the same parity is even.

Proof: Let $m, n \in \mathbb{Z}$ have the same parity. Then we have two cases:

Case ① Suppose both m and n are even. Then $m = 2p$ and $n = 2q$ for some $p, q \in \mathbb{Z}$. Then we have

$$m + n = 2p + 2q = 2(p + q)\ .$$

Since $\mathbb{Z}$ is a group, then $p + q \in \mathbb{Z}$. This implies that $m + n = 2k$ for some $k \in \mathbb{Z}$, and hence $m + n$ is even.

Case ② Suppose both m and n are odd. Then $m = 2p + 1$ and $n = 2q + 1$ for some $p, q \in \mathbb{Z}$. Then we have

$$\begin{aligned} m + n &= 2p + 1 + 2q + 1 \\ &= 2p + 2q + 2 = 2(p + q + 1)\ . \end{aligned}$$

Since $\mathbb{Z}$ is a group, then $p + q + 1 \in \mathbb{Z}$. This implies that $m + n = 2k$ for some $k \in \mathbb{Z}$, and $m + n$ is even.

$\therefore m + n$ is always even if m and n have the same parity. ■

Proposition 2: The sum of two integers with different parity is odd.

Proof: Let $m, n \in \mathbb{Z}$ with different parity. WLOG, suppose m is even and n is odd. Then $m = 2p$ and $n = 2q + 1$ for some $p, q \in \mathbb{Z}$. Then we have

$$m + n = 2p + 2q + 1 = 2(p + q) + 1 \ .$$

Since $\mathbb{Z}$ is a group, then $p + q \in \mathbb{Z}$. This implies that $m + n = 2k + 1$ for some $k \in \mathbb{Z}$.

$\therefore\ m + n$ is odd. ■

Now we will prove the claim that $\mathbb{J}_n$ is not a group for $n \geq 6$, as it is not closed, in the following cases:

Case ① Suppose $n \geq 12$ and n is even. Then we can write $J_{n,2}, J_{n,n-1} \in \mathbb{J}_n$ abstractly as follows:

$$J_{n,2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n-1 & n \\ 2 & 4 & 6 & 8 & \times & \dots & \times & \times \end{pmatrix}$$

and

$$J_{n,n-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots & n-1 & n \\ n-1 & n-2 & n & 2 & 5 & 9 & \dots & \times & \times \end{pmatrix} .$$

When composing them, we get

$$J_{n,n-1} \circ J_{n,2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n-1 & n \\ n-2 & 2 & 9 & \times & \times & \dots & \times & \times \end{pmatrix} .$$

If this permutation is $J_{n,k}$ for some $k \in \mathbb{N}$, then the first step implies that $k \equiv n - 2 \pmod{n}$, and hence $k = nq + (n - 2)$ for some $q \in \mathbb{Z}$. Since n is even, then both nq and $n - 2$ are even. By Proposition 1, k is even.

Notice that the third step implies that $k \equiv 7 \pmod{n-2}$, and hence $k = 7q + (n - 5)$ for some $q \in \mathbb{Z}$. Since n is even, then $(n - 2)q$ is even and 7 is odd. By Proposition 2, k is odd.

Yet, k cannot be both even and odd. Then $J_{n,n-1} \circ J_{n,2} \notin \mathbb{J}_n$, and hence $\mathbb{J}_n$ is not a group, as it is not closed.

Case ② Suppose $n \geq 23$ and n is odd. Then we can write $J_{n,2}, J_{n,n-1} \in \mathbb{J}_n$ abstractly as follows:

$$J_{n,2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n-1 & n \\ 2 & 4 & 6 & 8 & \times & \dots & \times & \times \end{pmatrix}$$

and

$$J_{n,n-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \dots & n-1 & n \\ n-1 & n-2 & n & 2 & 5 & 9 & 14 & 20 & \times & \dots & \times & \times \end{pmatrix} .$$

When composing them, we get

$$J_{n,n-1} \circ J_{n,2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n-1 & n \\ n-2 & 2 & 9 & 20 & \times & \dots & \times & \times \end{pmatrix} .$$

If this permutation is $J_{n,k}$ for some $k \in \mathbb{N}$, then the second step implies that $k \equiv 4 \pmod{n-1}$, and hence $k = (n-1)q + 4$ for some $q \in \mathbb{Z}$. Since n is odd, then $(n-1)$ is even and so is $(n-1)q$. Since 4 is even, then by Proposition 1, k is even. (The same result holds for the first and third steps.)

Notice that the fourth step implies that $k \equiv 11 \pmod{n-3}$, and hence $k = (n-3)q + 11$ for some $q \in \mathbb{Z}$. Since n is odd, then $(n-3)$ is even and so is $(n-3)q$. Since 11 is even, then by Proposition 2, k is odd.

Yet, k cannot be both even and odd. Then $J_{n,n-1} \circ J_{n,2} \notin \mathbb{J}_n$, and hence $\mathbb{J}_n$ is not a group, as it is not closed.

What we haven't shown is when $n \in \{6,7,8,9,10,11,13,15,17,19,21\}$. We will provide a counterexample to disprove closure for each $\mathbb{J}_n$ in these cases

Case ③ Suppose $n \in \{6,7,8,9,10,11,13,15,17,19,21\}$. We will consider three subcases.

Subcase ①: When $n \in \{6,8,10\}$, we claim that $J_{n,2} \circ J_{n,3} \notin \mathbb{J}_n$. We will show that $J_{6,2} \circ J_{6,3} \notin \mathbb{J}_6$, and the same procedure also works for $n = 8$ and $n = 10$.

If $n = 6$, then consider $J_{6,2}, J_{6,3} \in \mathbb{J}_6$. We know that

$$J_{6,2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 3 & 1 & 5 \end{pmatrix} = (1 \quad 2 \quad 4 \quad 3 \quad 6 \quad 5)$$

and

$$J_{6,3} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 2 & 5 & 1 \end{pmatrix} = (1 \quad 3 \quad 4 \quad 2 \quad 6) \quad .$$

When composing them, we will have the following:

$$J_{6,2} \circ J_{6,3} = (1 \quad 6 \quad 2 \quad 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 3 & 4 & 1 & 2 \end{pmatrix} \quad .$$

If this permutation is $J_{6,k}$ for some $k \in \mathbb{N}$, then the first step implies that $k \equiv 6 \pmod 6 = 0 \pmod 6$, which shows that k is even. The third step implies that $k \equiv 3 \pmod 4$, which shows that k is odd. Yet, k cannot be both even and odd. Then $J_{6,2} \circ J_{6,3} \notin \mathbb{J}_6$, and hence $\mathbb{J}_6$ is not a group, as it is not closed.

Subcase ②: When $n \in \{7,9,11,13,15\}$, we claim that $J_{n,3} \circ J_{n,2} \notin \mathbb{J}_n$. We will show that $J_{7,3} \circ J_{7,2} \notin \mathbb{J}_7$, and the same procedure also works for $n \in \{9,11,13,15\}$.

If $n = 7$, then consider $J_{7,3}, J_{7,2} \in \mathbb{J}_7$. We know that

$$J_{7,3} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 6 & 2 & 7 & 5 & 1 & 4 \end{pmatrix}$$

and

$$J_{7,2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 6 & 1 & 5 & 3 & 7 \end{pmatrix} .$$

When composing them, we get

$$J_{7,3} \circ J_{7,2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 1 & 3 & 5 & 2 & 4 \end{pmatrix} .$$

If this permutation is $J_{7,k}$ for some $k \in \mathbb{N}$, then the second step implies that $k \equiv 1 \pmod 6$, which shows that k is odd. The fourth step implies that $k \equiv 2 \pmod 4$, which shows that k is even. Yet, k cannot be both even and odd. Then $J_{7,3} \circ J_{7,2} \notin \mathbb{J}_7$, and hence $\mathbb{J}_7$ is not a group, as it is not closed.

Subcase ③: When $n \in \{17,19,21\}$, we claim that $J_{n,4} \circ J_{n,2} \notin \mathbb{J}_n$. We will show that $J_{17,4} \circ J_{17,2} \notin \mathbb{J}_n$, and the same procedure also works for $n = 19$ and $n = 21$.

If $n = 17$, then consider $J_{17,4}, J_{17,2} \in \mathbb{J}_{17}$. We know that

$$J_{17,4} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\ 4 & 8 & 12 & 16 & 3 & 9 & 14 & 2 & 10 & 17 & 7 & 1 & 13 & 11 & 15 & 6 & 5 \end{pmatrix}$$

and

$$J_{17,2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 1 & 5 & 9 & 13 & 17 & 7 & 15 & 11 & 3 \end{pmatrix}.$$

When composing them, we get

$$J_{17,4} \circ J_{17,2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 17 \\ 8 & 16 & 9 & 2 & \times & \dots & \times \end{pmatrix}$$

If this permutation is $J_{17,k}$ for some $k \in \mathbb{N}$, then the second step implies that $k \equiv 8 \pmod{16}$, which shows that k is even. The fourth step implies that $k \equiv 9 \pmod{14}$, which shows that k is odd. Yet, k cannot be both even and odd. Then $J_{17,4} \circ J_{17,2} \notin \mathbb{J}_{17}$, and hence $\mathbb{J}_{17}$ is not a group, as it is not closed.

Now we showed that $\mathbb{J}_n$ is not a group for $n \geq 6$, as it is not closed under composition. To sum up what we showed, we now know that

$\mathbb{J}_n = S_n$ for $n \in \{1,2,3\}$;

$\mathbb{J}_n = A_n$ for $n \in \{4,5\}$;

$\mathbb{J}_n$ is not a group for $n \geq 6$.

Activities

We will develop two Activities that capture the essence of the Josephus problem, and they are

Activity 1: Josephus Problem

Activity 2: Josephus Permutations

The goal of the first activity is to get students more familiar with the idea of a Josephus permutation, which builds students up for Activity 2 in a group theory class. Yet, the first activity can be used in a lower-level math class. For example, Activity 1 can be implemented in a number theory class where students are expected to look for pattern of the last survivor. Since the original context of the Josephus problem might be grim for students, teachers can choose to sugarcoat the context as long as the pattern is preserved.

Josephus Problem Activity

Main Objective: students will be able to

- be familiar with the Josephus problem in another context.
- come up with a formula to predict the pattern using mathematical thinking.
- realize that Josephus problem is a sequence/permutation.

Background: Students will be expected to have some prior knowledge of developing an algebraic expression to match up patterns as well as binary number systems.

Lesson Plan:

- Introduce the class to a simple paraphrasing of the Josephus Problem[2]. Students are asked to solve some Josephus Problems in order to be familiar with it.
- Split students in groups to work through some small examples of the Josephus Problem. Then collect the position of the winner with different circle sizes.
- Have students make any observations of the position of the winner with different n, and have them express their observations using mathematical language.
- Predict the position of the winner with some large value of n.

[2] To avoid the unpleasant context of the original Josephus problem (in which people die), we paraphrase in a different scenario.

Worksheet 1: Josephus Problem

Million Dollar Game

You are challenged to play the following game to win a million dollars:

Suppose there are 41 people in a circle labelled 1 – 41 clockwise. Starting from spot #1, every second person on the left is asked to step out from the circle one at a time clockwise until there is only ONE person left in this circle. This lucky person will go home with a million dollars.

Your goal is to win this game and become a millionaire!

1. Do you believe that there is a pattern for this game, or does it just purely depend on luck?

*In order to look for a pattern, we will start by considering a smaller circle. *

Let $n \in \mathbb{N}$ be the number of people in the circle. In the interest of maximizing our time, we will split up the task based on different sizes of the circle. Your data will be collected at the end for analysis in class, so be careful with the procedure.

(it is really hard to keep track of the steps when the size of the circle gets bigger…).

2. Please fill out the following table and be ready to share your finding with the class

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$j(n,2)$														
n	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$j(n,2)$														

*If your group has extra time, feel free to find out $j(n,2)$ when n is a large number (maybe $n = 41$?)

3. There are two observations worth noticing in the data. Please write down any observations you see.

We will define $j(n)$ as the position of the winner based on n. This is a function of $n \in \mathbb{N}$.

3. Can you write a rule for this function $j(n)$ using the two observations above?

Instead of deciding whether n is a power of 2, we want to have one formula that can gives us the position of the winner.

4. Can you combine two observations and write <u>one</u> formula for $j(n)$?

5. Notice that we didn't have $n = 1$ and $n = 2$ in our data table. What is $j(1)$ and $j(2)$ using our formula? Does the value make sense?

Now we have a formula that works for all $n \in \mathbb{N}$!

We will look at numbers in binary number system. Our goal is to discover a pattern in the binary system.

6. Please convert the data table from decimal system to the binary system below. What do you notice?

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$(n)_2$														
$(j(n))_2$														

Extension: We will prove the pattern discovered in the binary system.

Million Dollar Game - Key

You are challenged to play the following game to win a million dollars:

Suppose there are 41 people in a circle labelled 1 – 41 clockwise. Starting from spot #1, every second person on the left is asked to step out from the circle one at a time clockwise until there is only ONE person left in this circle. This lucky person will go home with a million dollars.

Your goal is to win this game and become a millionaire!

1. Do you believe that there is a pattern for this game, or does it just purely depend on luck?

Of course, there is a pattern. This is a math class and mathematicians don't do things purely dependent on luck.

*In order to look for a pattern, we will start by considering a smaller circle. *

Let $n \in \mathbb{N}$ be the number of people in the circle. In the interest of maximizing our time, we will split up the task based on different sizes of the circle. Your data will be collected at the end for analysis in class, so be careful with the procedure.

(it is really hard to keep track of the steps when the size of the circle gets bigger…).

2. Please fill out the following table and be ready to share your finding with the class

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$j(n,2)$	3	1	3	5	7	1	3	5	7	9	11	13	15	1
n	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$j(n,2)$	3	5	7	9	11	13	15	17	19	21	23	25	27	29

*If your group has extra time, feel free to find out $j(n,2)$ when n is a large number (maybe $n = 41$?)

3. There are two observations worth noticing in the data. Please write down any observations you see.

Two observations are:

1. when $n = 2^k$ for some $k \geq 2$, the position of the winner is always 1.

2. when $n \in [2^k, 2^{k+1})$ for some $k \geq 2$, the position of the winners for each n forms a sequence of odd numbers.

*Students will need these two observations to come up with a formula for the game, so make sure students can see these two observations before moving on.

We will define $j(n)$ as the position of the winner based on n. This is a function of $n \in \mathbb{N}$.

3. Can you write a rule for this function $j(n)$ using the two observations above?

Observation 1 states that $j(n) = 1$ if $n = 2^k$ for some $k \geq 2$; and

Observation 2 states that $j(n) = 2l + 1$ if $n = 2^k + l$ for some $k \geq 2$ and $l \in [0,2^k)$.

Hence, we can write $j(n)$ as a piecewise function below:

$$j(n) = \begin{cases} 1 & \text{if } n = 2^k \text{ for some } k \geq 2 \\ 2l + 1 & \text{if } n = 2^k + l \text{ for some } k \geq 2, l \in [0,2^k) \end{cases}.$$

Instead of deciding whether n is a power of 2, we want to have one formula that can gives us the position of the winner.

4. Can you combine two observations and write <u>one</u> formula for $j(n)$?

The idea is to reduce n, l, k to one independent variable. The formula is $j(n) = 2\left(n - 2^{\lfloor \log_2 n \rfloor}\right) + 1$. The detailed work can be found on page 9 to page 10.

5. Notice that we didn't have $n = 1$ and $n = 2$ in our data table. What is $j(1)$ and $j(2)$ using our formula? Does the value make sense?

Using the formula, $j(1) = j(2) = 1$. The value makes sense. If there is only one person in the circle, it automatically makes him the winner by the rule of the game; if there are two people in the circle, spot #1 will ask spot #2 to step out of the circle, which results in only one person in the circle, which is spot #1.

Now we have a formula that works for all $n \in \mathbb{N}$!

We will look at numbers in binary number system. Our goal is to discover a pattern in the binary system.

6. Please convert the data table from decimal system to the binary system below. What do you notice?

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$(n)_2$	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110
$\left(j(n)\right)_2$	1	1	11	1	11	101	111	1	11	101	111	1001	1011	1101

The pattern is the following: If we take the leading 1 in $(n)_2$ and put it in the last place, we get $\left(j(n)\right)_2$.

Extension: We will prove the pattern discovered in the binary system.

The proof of this pattern can be found on pages 11-12.

Josephus Permutations Activity

Objectives: Students will be able to

- Understand what is meant by a permutation and recognize the permutation group.
- Work with a set of permutations whose values are given by a recursive process, rather than an explicit algebraic rule.
- Apply the subgroup test to determine whether a set is a group (subgroup of a symmetric group).

Background:

Students should have some prior knowledge of permutation groups and subgroup tests. That also implies that students should be familiar with the group axioms. It is recommended that students have seen the alternating groups. This activity is meant to provide students with an application / example of permutation groups using the famous puzzle in mathematics called the Josephus Problem.

Lesson Plan:

Introduction

- Introduce the class to a simple paraphrasing of the Josephus Problem[3]. Students are asked to solve some Josephus Problems in order to be familiar with it.
- Demonstrate some examples and non-examples of Josephus permutations.

Abstract the game

- Instead of working with actual numbers $n, k \in \mathbb{N}$, we consider variables n and k.
- Figure out the number of elements in this set of permutations for a fixed n.

[3] To avoid the unpleasant context of the original Josephus problem (in which people die), we paraphrase in a different scenario.

Subgroup Test

- Determine whether the permutations form a subset of a symmetric group.
- Working through the first example: $n = 4$ and $k \in \mathbb{N}$.
- Working through the second example: $n = 6$ and $k \in \mathbb{N}$.
- Apply the subgroup test to the set of Josephus permutations for $n = 6$.
- (time permitting) Examine the Josephus permutations for $n = 5$ (or this could be an extension).

Conclusion / Extension

- Conclusion: Classify the Josephus permutations for $n \in \{1,2,3,4\}$.
- Extension: Examine the Josephus permutations for $n = 5$.
- Extension: Examine the Josephus permutations for $n > 6$.

Teacher note

- The scenario of the Josephus permutation can be altered so that students will be interested in the problem. Also please do not mention the term "Josephus Problem" throughout the lecture so that students won't Google it.
- The activity requires a lot of counting. It will be beneficial if students can work in groups so that they can split up some tasks.
- Some part of the activity requires extensive skills of number theory (showing closure and inverses). Hence, students are strongly encouraged to work with specific examples of the Josephus permutation rather than solving it abstractly.
- The following link might be useful to obtain a Josephus permutation with a fixed n and k:

  http://webspace.ship.edu/deensley/mathdl/joseph.html

Regarding the Josephus subgroup for $n = 5$…

- The Josephus subgroup for $n = 5$ requires so much calculation / enumeration that students in the undergraduate level are unlikely to complete it in an hour class in addition to other activities. Hence, it could be a great extension for students to explore in their spare time.
- If the class size is large (30 or more), the Josephus subgroup for $n = 5$ can be classified using enumeration. Students can split up the task to find all 60 Josephus permutations for $n = 5$ and compare it to A_5.

Regarding the Josephus permutation set for $n \geq 6$:

- Coming up with a counterexample to disprove closure is not easy. Some hints can be given to students in the interest of saving class time. The counterexamples are provided as follows:
- $J_{n,n-1} \circ J_{n,2} \notin \mathbb{J}_n$ when $n \geq 12$ and n is even.
- $J_{n,n-1} \circ J_{n,2} \notin \mathbb{J}_n$ when $n \geq 23$ and n is odd.
- $J_{n,2} \circ J_{n,3} \notin \mathbb{J}_n$ when $n \in \{6,8,10\}$.
- $J_{n,3} \circ J_{n,2} \notin \mathbb{J}_n$ when $n \in \{7,9,11,13,15\}$.
- $J_{n,4} \circ J_{n,2} \notin \mathbb{J}_n$ when $n \in \{17,19,21\}$.

<u>Million Dollar Jackpot Game</u>

We will call the permutation group the jackpot permutation group

You are challenged to play the following game to win a million dollars:

Suppose there are 41 people in a circle labelled 1 – 41 clockwise. Starting from spot #1, every second person on the left is asked to step out from the circle one at a time clockwise until there is only ONE person left in this circle. This lucky person will go home with a million dollars.

Your goal is to win this game and become a millionaire!

Prequal: Explore this game

1. Do you believe that there is a pattern for this game, or does it just purely depend on luck?

*In order to solve for this pattern, we will start by looking at the smaller circle. *

2. Play through the game with a smaller number of people in the circle. Pick your favorite number as the number of people in the circle and record the number of the last spot.

(Bonus: Can you see a pattern and predict which spot you should stand to win a million?)

*I am sure you realize that the sequence of people stepping out forms a permutation. This is what we will focus on! *

Part I: Abstract the Game

Let $n \in \mathbb{N}$ be the number of people in a circle and $k \in \mathbb{N}$ be the number skipped at each step. Then the order of people stepping out of the circle each time forms a permutation (We will call it a Jackpot permutation). Mathematicians like to make everything abstract so we will work through the abstraction.

1. Write a Jackpot permutation of your choice of n and k.

Which notation did you use? Standard notation or cyclic notation. Which one do you think is better in this case?

Notice that the larger n gets, the longer the permutation can get, which adds more difficulty. So, we will fix n and let k be any arbitrary natural numbers.

3. Given a permutation, can you tell whether this permutation can be generated by the rule of the game? We will look at a few examples:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

4. Can you guess how many of such permutations they are for a fixed n?

Part II: Permutation Groups Example 1

Now we move to some group theory stuff.

Consider $n = 4$. We will use $\mathbb{J}_4$ to denote the set of all permutations generated by the rule with $n = 4$.

I claim that $\mathbb{J}_4$ is a group.

Proof: We will show that $\mathbb{J}_4$ is a group by showing it satisfies the group axioms.

$\mathcal{G}1$: Is the operation in $\mathbb{J}_4$ associative?

$\mathcal{G}2$: Is there an identity in $\mathbb{J}_4$?

[Hint: How can you modify the rule so that it corresponds to the identity permutation?]

$\mathcal{G}3$: For each permutation in $\mathbb{J}_4$, is there an inverse in $\mathbb{J}_4$?

If you can't solve this right away, you can just take my word and assume inverses exist for each element in $\mathbb{J}_4$ for now!

Question: Can I conclude that $\mathbb{J}_4$ is a group? What am I missing? Show the missing piece!

Now we can conclude that $\mathbb{J}_4$ is a group!

Consider the size of $\mathbb{J}_n$ and S_n. We know that $|\mathbb{J}_n| = lcm[1,2,\dots,n]$ and $|S_n| = n!$. Compare these two numbers. What do you notice?

Using this pattern, what can you say about $\mathbb{J}_1$, $\mathbb{J}_2$, $\mathbb{J}_3$, and $\mathbb{J}_5$.

*We will put aside $\mathbb{J}_5$ now and take a look at $\mathbb{J}_6$, since $|\mathbb{J}_6| < A_6$, then we cannot conjecture that $\mathbb{J}_6$ is a group for now. *

Part III: Permutation Groups Example 2

A natural question to ask is whether $\mathbb{J}_n$ is a group for ALL $n \in \mathbb{N}$. We will take a look at $\mathbb{J}_6$.

What is your claim? Do you believe that $\mathbb{J}_6$ is a group? Justify your response.

(That being said, if you believe $\mathbb{J}_6$ is a group, then prove it; if you think $\mathbb{J}_6$ is not a group, then why it isn't).

Extension: We left out $\mathbb{J}_5$. If you are interested in determining whether $\mathbb{J}_n$ forms a group for any $n \in \mathbb{N}$, there are three questions that might be able to help.

- Is $\mathbb{J}_5$ isomorphic to A_5?
- Is $\mathbb{J}_n$ a group when $n > 6$?

See if you can answer these questions.

Million Dollar Jackpot Game - Key

We will call the permutation group the jackpot permutation group

You are challenged to play the following game to win a million dollars:

Suppose there are 41 people in a circle labelled 1 – 41 clockwise. Starting from spot #1, every second person on the left is asked to step out from the circle one at a time clockwise until there is only ONE person left in this circle. This lucky person will go home with a million dollars.

Your goal is to win this game and become a millionaire!

Prequal: Explore this game

1. Do you believe that there is a pattern for this game, or does it just purely depend on luck?

Of course, there is a pattern. This is a math class and mathematicians don't do things purely dependent on luck.

*In order to solve for this pattern, we will start by looking at the smaller circle. *

2. Play through the game with a smaller number of people in the circle. Pick your favorite number as the number of people in the circle and record the number of the last spot.

(Bonus: Can you see a pattern and predict which spot you should stand to win a million?)

We will provide a table of the number of people and the number of the last spot in the circle below:

n	$J(n)$
1	1
2	1
3	3
4	1
5	3
6	5
7	7
8	1
9	3
10	5
11	7
12	9
13	11
14	13
15	15
16	1
17	3
18	5
19	7
20	9
21	11
22	13
23	15
24	17
25	19
26	21
27	23
28	25
29	27
30	29
31	31
32	1
33	3
34	5
35	7
36	9
37	11
38	13
39	15
40	17
41	19

The winning spot is spot #19.

*I am sure you realize that the sequence of people stepping out forms a permutation. This is what we will focus on! *

Part I: Abstract the Game

Let $n \in \mathbb{N}$ be the number of people in a circle and $k \in \mathbb{N}$ be the number skipped at each step. Then the order of people stepping out of the circle each time forms a permutation (We will call it a Jackpot permutation). Mathematicians like to make everything abstract so we will work through the abstraction.

1. Write a Jackpot permutation of your choice of n and k.

An example will be the permutation of $n = 5$ and $k = 2$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix} \qquad \text{OR} \qquad (1 \quad 2 \quad 4 \quad 5 \quad 3)$$

Which notation did you use? Standard notation or cyclic notation. Which one do you think is better in this case?

Standard notation is better since it indicates the total number of people (n), whereas the cyclic notation might not express n explicitly (there might be more cycles consisting of one element in the end that didn't get written out).

Also, it is clear to see the sequence of when people step out of the circle in the standard notation.

Notice that the larger n gets, the longer the permutation can get, which adds more difficulty. So, we will fix n and let k be any arbitrary natural numbers.

3. Given a permutation, can you tell whether this permutation can be generated by the rule of the game? We will look at a few examples:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

The first permutation can be generated by the rule. It is when $n = 4$ and $k = 3$.

The third permutation can be generated by the rule. It is when $n = 4$ and $k = 7$.

The second permutation CANNOT be generated by the rule because:

The first spot from 1 to 2 indicates that $k \equiv 2 \pmod 4$;

The second spot from 3 to 3 (remember that 2 is stepped out) indicates that $k \equiv 1 \pmod 3$;

The third spot from from 4 to 1 indicates that $k \equiv 1 \pmod 2$

The second spot indicates k is even, but the third spot indicates k is odd. There is no such $k \in \mathbb{N}$.

4. Can you guess how many of such permutations they are for a fixed n?

Instead of having them prove that the number of such permutations is $lcm[1,2,\dots,n]$, we will have students take a guess and then point out this fact (to avoid too much time spent on number theory stuff).

Part II: Permutation Groups Example 1

Now we move to some group theory stuff.

Consider $n = 4$. We will use $\mathbb{J}_4$ to denote the set of all permutations generated by the rule with $n = 4$.

I claim that $\mathbb{J}_4$ is a group.

Proof: We will show that $\mathbb{J}_4$ is a group by showing it satisfies the group axioms.

$\mathcal{G}1$: Is the operation in $\mathbb{J}_4$ associative?

Recall that permutations are just functions. In order words, show that function compositions are associative.

Let f, g, h be functions with corresponding domains and codomains and $x \in D$ where D is the domain of h. Then

$$\big((f \circ g) \circ h\big)(x) = (f \circ g)\big(h(x)\big) = f\left(g\big(h(x)\big)\right)$$

and

$$\big(f \circ (g \circ h)\big)(x) = f\big((g \circ h)(x)\big) = f\left(g\big(h(x)\big)\right).$$

Notice that $\big((f \circ g) \circ h\big)(x) = f\left(g\big(h(x)\big)\right) = \big(f \circ (g \circ h)\big)(x)$ for any arbitrary $x \in D$. Then $(f \circ g) \circ h = f \circ (g \circ h)$.

Therefore, function composition is associative, and hence, the operation in a permutation group is associative.

$\mathcal{G}2$: Is there an identity in $\mathbb{J}_4$?

[Hint: How can you modify the rule so that it corresponds to the identity permutation?]

Consider $J_1 = (1)$, which is the permutation of $k = 1$. This is the identity permutation.

Let $J \in \mathbb{J}_5$. Then $(1)J = J(1) = J$.

$\mathcal{G}3$: For each permutation in $\mathbb{J}_4$, is there an inverse in $\mathbb{J}_4$?

The pattern is rather complicated. The inverse of each element in $\mathbb{J}_4$ is presented in **Appendix A**.

Students should be considering each specific element and determine whether their inverse is in $\mathbb{J}_4$.

*This question is intended to encourage students to write out each element in $\mathbb{J}_4$ for later comparison to A_4. *

If you can't solve this right away, you can just take my word and assume inverses exist for each element in $\mathbb{J}_4$ for now!

Question: Can I conclude that $\mathbb{J}_4$ is a group? What am I missing? Show the missing piece!

We showed that $\mathbb{J}_4$ satisfies the group axioms ASSUMING composition in $\mathbb{J}_4$ is a binary operation. Hence, we need to show that the composition is a binary operation.

We need to show that $\mathbb{J}_4$ is closed under composition.

We can show closure by eliminating k's according to each spot in the permutation. Since $|\mathbb{J}_4| = lcm\{1,2,3,4\} = 12$, then the list is short and manageable. If students having trouble using this method, we propose a different method as follows:

Consider that $|\mathbb{J}_4| = 12 = \frac{24}{2} = \frac{4!}{2} = \frac{|S_4|}{2}$. We can conjecture that $\mathbb{J}_4 = A_4$ (this is a plausible conjecture). Once students wrote out all the elements in $\mathbb{J}_4$, they can realize that $\mathbb{J}_4$ consists of ALL the even permutations (See **Appendix A**), and hence $\mathbb{J}_4 = A_4$. Showing A_4 is a subgroup of S_4 proves that $\mathbb{J}_4$ is a group (by transitivity of isomorphisms).

Students are encouraged to show that A_4 is a group by finite subgroup test

Now we can conclude that $\mathbb{J}_4$ is a group!

Consider the size of $\mathbb{J}_n$ and S_n. We know that $|\mathbb{J}_n| = lcm\{1,2,\dots,n\}$ and $|S_n| = n!$. Compare these two numbers. What do you notice?

We have the following table to record their values:

n	1	2	3	4	5	6	7
$\lvert\mathbb{J}_n\rvert$	1	2	6	12	60	60	420
$\lvert S_n\rvert$	1	2	6	24	120	720	5040

Students should notice that $|\mathbb{J}_n| = |S_n|$ when $n \in \{1,2,3\}$, $|\mathbb{J}_n| = \frac{|S_n|}{2}$ when $n \in \{4,5\}$, and $|\mathbb{J}_n| < \frac{|S_n|}{2}$ when $n \geq 6$.

Using this pattern, what can you say about $\mathbb{J}_1$, $\mathbb{J}_2$, $\mathbb{J}_3$, and $\mathbb{J}_5$.

Notice that $\mathbb{J}_n \subseteq S_n$ and $|\mathbb{J}_n| = |S_n|$ when $n \in \{1,2,3\}$. This implies that

$$\mathbb{J}_1 = S_1, \mathbb{J}_2 = S_2, \text{ and } \mathbb{J}_3 = S_3.$$

We showed that $\mathbb{J}_4 \cong A_4$, which makes $\mathbb{J}_4$ a group (We did so by noticing that $|\mathbb{J}_4| = |A_4|$ and make the conjecture)

Since $|\mathbb{J}_5| = |A_5|$, then we can make the plausible conjecture that $\mathbb{J}_5 \cong A_5$.

See **Appendix B** that all permutations in $\mathbb{J}_5$ are even permutation, and hence $\mathbb{J}_5 \cong A_5$.

*We will put aside $\mathbb{J}_5$ now and take a look at $\mathbb{J}_6$, since $|\mathbb{J}_6| < A_6$, then we cannot conjecture that $\mathbb{J}_6$ is a group for now. *

Part III: Permutation Groups Example 2

A natural question to ask is whether $\mathbb{J}_n$ is a group for ALL $n \in \mathbb{N}$. We will take a look at $\mathbb{J}_6$.

What is your claim? Do you believe that $\mathbb{J}_6$ is a group? Justify your response.

(That being said, if you believe $\mathbb{J}_6$ is a group, then prove it; if you think $\mathbb{J}_6$ is not a group, then why it isn't).

$\mathbb{J}_6$ is NOT a group as it is NOT closed. A counterexample will be $J_{6,2} \circ J_{6,3}$. Notice that

$$J_{6,2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 3 & 1 & 5 \end{pmatrix} = (1 \quad 2 \quad 4 \quad 3 \quad 6 \quad 5)$$

and

$$J_{6,3} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 2 & 5 & 1 \end{pmatrix} = (1 \quad 3 \quad 4 \quad 2 \quad 6)$$

Then when composing these two permutations, when get

$$J_{6,2} \circ J_{6,3} = (1 \quad 2 \quad 4 \quad 3 \quad 6 \quad 5)(1 \quad 3 \quad 4 \quad 2 \quad 6)$$

$$= (1 \quad 6 \quad 2 \quad 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 3 & 4 & 1 & 2 \end{pmatrix}.$$

The first step is from 1 to 6. Then k is divisible by 6, and hence $k \equiv 0 \pmod 6$, which implies that k is even.

The second step is from 1 to 5. Then k is divisible by 5, and hence $k \equiv 0 \pmod 5$.

The third step is from 1 to 3. Then k is divisible by 3, and hence $k \equiv 3 \pmod 4$, which implies that k is odd.

Notice that k cannot be both even and odd. Hence, this permutation CANNOT be generated by the rule. We can conclude that $\mathbb{J}_6$ is NOT closed. It fails the finite subgroup test, and hence, $\mathbb{J}_6$ is NOT a group.

Extension: We left out $\mathbb{J}_5$. If you are interested in determining whether $\mathbb{J}_n$ forms a group for any $n \in \mathbb{N}$, there are three questions that might be able to help.

- Is $\mathbb{J}_5$ isomorphic to A_5?
- Is $\mathbb{J}_n$ a group when $n > 6$?

See if you can answer these questions.

Implementation and Reflection

The second lesson plan presented in the previous section was implemented in a 300-level group theory class (MTH 344) taught by Dr. Julie Bracken. After communicating with Dr. Bracken about how much material in group theory the class has gone over, I decided to cut off unnecessary material in number theory and focus more on the group theory, like subgroups, permutation groups, and alternating groups. The duration of the class is 1 hour and 5 minutes. A concern I had before implementing the lesson plan was the length of the activity. Thankfully, Dr. Bracken agreed to let me get started using the last 10 minutes of the previous class to introduce the rule of the game (labeled prequel in the lesson plan) so students could get started on the Josephus permutation (labeled part I, II, and III in the lesson plan) on the implementation day, which gave me 1 hour and 15 minutes in total for the implementation. The estimated time spent on each part of the activity is listed below:

Section	**Duration**
Prequel: Explore the Game	20 minutes
Part I: Abstract the Game	15 minutes
Part II: Permutation Group Example 1	25 minutes
Part III: Permutation Group Example 2	15 minutes

Because the primary goal of the activity is to provide students with an example of a subgroup of a symmetric group using an actual scenario, more time was spent on part II of the activity, where students were asked to determine whether $\mathbb{J}_4$ is a group using what they have learned about groups, subgroups, and permutation groups. That being said, the prequel and part I of the activity was meant to build students up so they have all necessary information to tackle on part II.

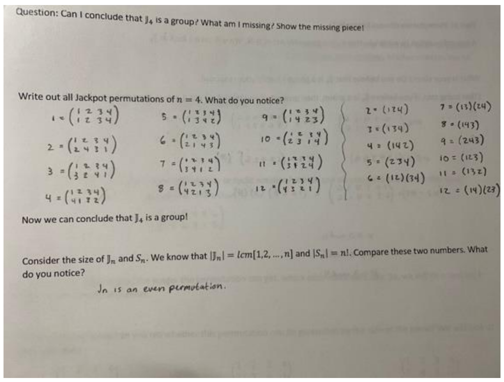
The actual implementation did not go well as I predicted. We will discuss the difficulty of each section.

In the prequel, which was implemented in the last 10 minutes of the previous lecture, I mostly went through the rule of the game, hoping students will have a day to process this information, as the rule might be a little confusing at the first glimpse. A worksheet, as well as a detailed instruction, was given to students to study in their own time. Students did not raise any questions in the last 10 minutes of the previous lecture, which gave me a false impression that students could comprehend the rule easily. In the first 10 minutes of the lecture in the implementation day, I had hoped to start in on part I. Yet, quite a few students expressed their confusion about the rule of the game. Given that this was a crucial step of the activity, another 10 minutes was spent to ensure students fully understood it.

In part I of the activity, a difficulty I observed in class was the confusion about k. We defined k as the number of people skipped in each step. In the original million-dollar question, "every person on the left" implies that every second person is asked to step out, indicating that $k = 2$. Some students had trouble playing the game with a different k. Thankfully, with the help of Dr.Bracken, students were able to overcome this difficulty, and before beginning Part I, most students were able to play through the game with different values of k. Also, because the proof of $|\mathbb{J}_n| = lcm[1,2,\dots,n]$ involves some number theory, which might have caused more confusion, I chose to state this as a fact without proof. A student in class raised a doubt as to why it is true. Since this is not a number theory class, I chose to walk her through the idea only briefly.

In part II of the activity, there seemed to be too much information packed in. To review the group axioms, I chose to have students prove that $\mathbb{J}_4$ is a group using the axioms. Proving closure and inverses, again, involved number theory, which is not the topic of this course, so a different strategy was used, which was listing out all the 12 elements in $\mathbb{J}_4$. Since the class is small (estimated 14 students), I planned on splitting up the task so that each student only had to write out a few elements of $\mathbb{J}_4$ on the board.

Unfortunately, students chose to work in small groups (not on the board). Also, Dr. Bracken had only mentioned the term “alternating group” a few times before, so students might not have discovered that all elements in $\mathbb{J}_4$ are even on their own. Also, a short proof of the finite subgroup test was given with a lecture in class so that students could use this test in the later activity. Some students work is shown below. As their work suggested, some would be able to write out all 12 elements in $\mathbb{J}_4$.

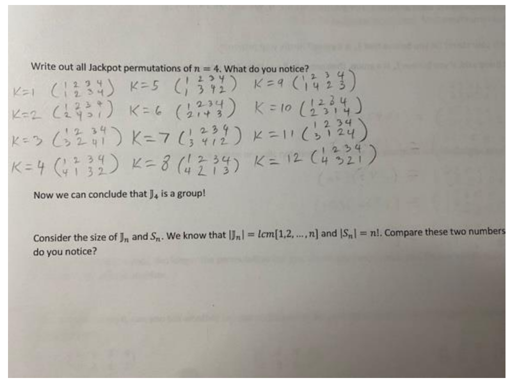

Question: Can I conclude that $\mathbb{J}_4$ is a group? What am I missing? Show the missing piece!

Write out all Jackpot permutations of $n = 4$. What do you notice?

Now we can conclude that $\mathbb{J}_4$ is a group!

Consider the size of $\mathbb{J}_n$ and S_n. We know that $|\mathbb{J}_n| = lcm[1,2,\ldots,n]$ and $|S_n| = n!$. Compare these two numbers. What do you notice?

Jn is an even permutation.

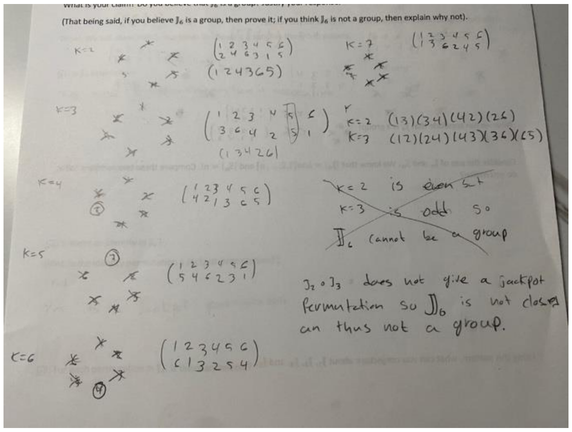

Write out all Jackpot permutations of $n = 4$. What do you notice?

Now we can conclude that $\mathbb{J}_4$ is a group!

Consider the size of $\mathbb{J}_n$ and S_n. We know that $|\mathbb{J}_n| = lcm[1,2,\ldots,n]$ and $|S_n| = n!$. Compare these two numbers. What do you notice?

Figure 1 Selected Students work in Part II

In part III of the activity, there was a huge misunderstanding derived from the last part of the activity. The goal was to have students either prove $\mathbb{J}_6$ is closed under composition (and hence $\mathbb{J}_6$ is a group by the finite subgroup test), or provide a counterexample to show that $\mathbb{J}_6$ is not closed (and hence $\mathbb{J}_6$ is not a group). In the interest of saving time, a hint was given to students to examine $J_{6,2}$ and $J_{6,3}$. The hope was for students to discover that $J_{6,2} \circ J_{6,3} \notin \mathbb{J}_6$. Yet, because alternating groups were mentioned in $\mathbb{J}_4$, some students chose only to show that $J_{6,2}$ being even and $J_{6,3}$ being odd implies that $\mathbb{J}_6$ is not a group, which is insufficient (as shown below in students work).



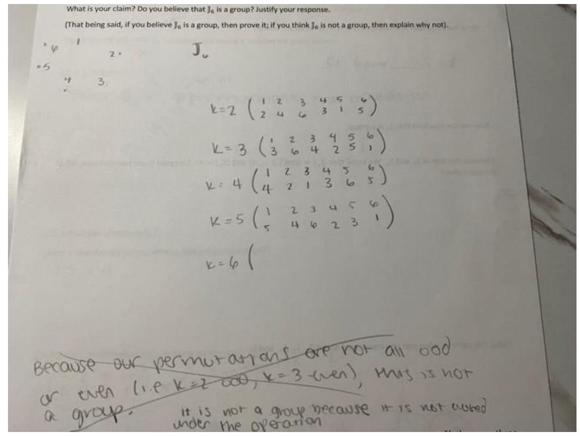


Figure 2 Selected Students work in Part III

Overall, the main difficulty about my attempted implementation was the time. There was too much material/information packed in the activity to implement in one class period. Also, there were too many questions in the handout that students knew before the implementation (for example, students knew that function composition is associative). To save time, some questions can be cut to condense the handout. Ideally, more time should be spent walking through the prequel and part I of the activity so that students can feel more comfortable working through the rest of the activity, as one student pointed out that he/she felt behind throughout the whole activity.

There is one more thing worth noticing during the implementation. As mentioned earlier, the Josephus problem can be programmed easily on a computer to find the position of the last survivor, as well as the sequence of people being killed. A student in class programmed the game to solve the entire activity. It occurred to me that this activity could also be a good exercise for computer science and students can see how computer science can be used to solve math problems.

Conclusion

The Josephus problem is a well-known topic in mathematics and computer science, and it is related to a certain counting game. Most of the existing curriculum that has been developed to make use of this problem involves only exploring the recurrence relation to compute the position of the last survivor. In this project, we have also developed a lesson plan in the same topic, but we have added some extra material that allows students to discover a fun pattern using binary numbers.

The main section of the activity concerns the set of Josephus permutations with a fixed n, denoted $\mathbb{J}_n$. We showed that $\mathbb{J}_n$ is a group when $n \in \{1,2,3,4,5\}$ and $\mathbb{J}_n$ is NOT a group when $n \geq 6$. The activity is meant to provide students with an example of permutation groups in a specific context. Students are led to develop necessary knowledge in group theory (including the finite subgroup test, permutation groups, alternating groups, etc.) to work on the activity.

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Appendix A Elements in $\mathbb{J}_4$

The elements in $\mathbb{J}_4$ are recorded in the following table in the cyclic notation:

$\boldsymbol{k}$	$\boldsymbol{J_{4,k}}$
1	(1)
2	$(1\ 2\ 4)(3)$
3	$(1\ 3\ 4)(2)$
4	$(1\ 4\ 2)(3)$
5	$(1)(2\ 3\ 4)$
6	$(1\ 2)(3\ 4)$
7	$(1\ 3)(2\ 4)$
8	$(1\ 4\ 3)(2)$
9	$(1)(2\ 4\ 3)$
10	$(1\ 2\ 3)(4)$
11	$(1\ 3\ 2)(4)$
12	$(1\ 4)(2\ 3)$

* $J_{4,k}$ represents the permutation generated by $n = 4$ and $k \in \mathbb{N}$.

* All the fixed points are presented in the form of a 1-cycle.

* The inverse pairs in $\mathbb{J}_4$ are : $(J_{4,2}, J_{4,4})$, $(J_{4,3}, J_{4,8})$, $(J_{4,5}, J_{4,9})$, $(J_{4,10}, J_{4,11})$, $(J_{4,6}, J_{4,6})$, $(J_{4,7}, J_{4,7})$, $(J_{4,12}, J_{4,12})$.

Appendix B Elements in $\mathbb{J}_5$

The elements in $\mathbb{J}_5$ are recorded in the following table in the cyclic notation:

$\boldsymbol{k}$	$\boldsymbol{J_{5,k}}$
1	(1 2 4 5 3)
2	(1 3 5 4 2)
3	(1 4 2 3 5)
4	(1 5 2)(3)(4)
5	(1)(2 3)(4 5)
6	(1 2 5 4 3)
7	(1 3 5)(2)(4)
8	(1 4)(2 5)(3)
9	(1 5 4)(2)(3)
10	(1)(2 4 3)(5)
11	(1 2)(3 5)(4)
12	(1 3 5 2 4)
13	(1 4 2)(3)(5)
14	(1 5)(2 3)(4)
15	(1)(2 5 3)(4)
16	(1 2 3 5 4)
17	(1 3 4 2 5)
18	(1 4 5)(2)(3)
19	(1 5 3 2 4)
20	(1)(2)(3 5 4)
21	(1 2 4 3 5)
22	(1 3 4 5 2)
23	(1 4)(2 3)(5)
24	(1 5 4 3 2)
25	(1)(2 3 5)(4)
26	(1 2 5 3 4)
27	(1 3 4)(2)(5)
28	(1 4 3 2 5)
29	(1 5 3)(2)(4)
30	(1 2 4 5 3)
31	(1)(2 4)(3 5)
32	(1 2)(3 4)(5)
33	(1 3 2 4 5)
34	(1 4 5 3 2)
35	(1 5 4 2 3)
36	(1)(2 5)(3 4)
37	(1 2 3 4 5)
38	(1 3 2 5 4)
39	(1 4 3)(2)(5)
40	(1 5 2 4 3)
41	(1)(2)(3 4 5)
42	(1 2 4)(3)(5)
43	(1 3 2)(4)(5)
44	(1 4 5 2 3)
45	(1 5 3 4 2)
46	(1)(2 3 4)(5)
47	(1 2 5)(3)(4)
48	(1 3)(2)(4 5)
49	(1 4 2 5 3)
50	(1 5)(2)(3 4)
51	(1)(2 4 5)(3)
52	(1 2)(3)(4 5)
53	(1 3)(2 4)(5)
54	(1 4 3 5 2)
55	(1 5 2 3 4)
56	(1)(2 5 4)(3)
57	(1 2 3)(4)(5)
58	(1 3)(2 5)(4)
59	(1 4)(2)(3 5)
60	(1 5)(2 4)(3)

* $J_{5,k}$ represents the permutation generated by $n = 5$ and $k \in \mathbb{N}$.

* All the fixed points are presented in the form of a 1-cycle.